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LETTER TO THE EDITOR

# Metric properties of fractal lattices

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**Abstract.** We study the connectivity dimension $\hat{d}$ of fractal lattices viewed as networks (graphs) of sites and (constant length) bonds. Two examples are investigated in detail: the 2D Sierpinski gasket and 2D infinite percolation clusters on square and triangular lattices. In the first case $\hat{d}$ is shown to coincide with the fractal dimension, whereas $\hat{d} = 1.72 \pm 0.02$ appears as a universal exponent for percolation clusters in two dimensions. Considered as an intrinsic parameter, the connectively dimenion $\hat{d}$ is compared with other intrinsic and extrinsic characteristic parameters of fractal lattices. In particular we argue that $\tilde{d} \leqslant \hat{d} \leqslant \bar{d}$ holds on fractal lattices in general ($\bar{d}$ = fractal dimension, $\tilde{d}$ = spectral dimension).

It is now recognised that there are many self-similar structures (fractals) in nature and many ways to model them (Mandelbrot 1982). Most of the attention so far has been focused on characterising the geometrical properties of fractals. The fractal dimension $\bar{d}$ hence emerges as a first operative measure of the fractal geometry. Recently, it has been shown (Alexander and Orbach 1982, Rammal and Toulouse 1983) that simple physical problems on fractal lattices (spectrum of low energy excitations, classical diffusion, ...) are governed by another dimension: the spectral dimension $\tilde{d}$. For instance, $\tilde{d} \simeq \frac{4}{3}$ and $1 < \bar{d} \leqslant 4$ for percolation clusters at all Euclidean dimensions $d \geqslant 2$. In contrast, for the Sierpinski gaskets, $\tilde{d}$ and $\bar{d}$ both depend on $d$: $\bar{d} = \ln(d+1)/\ln(2)$ and $\tilde{d} = 2\ln(d+1)/\ln(d+3)$ (see e.g. Angles d'Auriac *et al* 1983). Very recently, it was realised that $\tilde{d}$ is an intrinsic parameter, independent of the space in which the fractal is embedded, whereas $\bar{d}$ depends on this embedding (Rammal *et al* 1984). One reason for this distinction is that $\tilde{d}$ provides the proper counting of closed paths, whereas $\bar{d}$ is appropriate for the counting of points (or masses) at a given length scale. For instance, it has been shown that the self-avoiding walk (SAW) statistics on a fractal lattice is governed by another intrinsic exponent, independent of the embedding space, which depends on $\tilde{d}$, and possibly on other parameters, as soon as $\tilde{d} \leqslant 4$ (Rammal *et al* 1984). Moreover, it was shown that the spectral and fractal dimensions of the backbone are not sufficient to determine the properties of SAWs in general. Therefore, it is tempting to speculate that $\tilde{d}$ is just the first of a hierarchy of intrinsic indices, controlling more and more specific properties of fractal lattices. In this Letter, we investigate another intrinsic property of fractal structures, viewed as infinite lattice graphs: the connectivity dimension.

Basically, the connectivity dimension deal with the notion of distance between nodes implied by the lattice graph structure of the fractal. In general, the distance $d(x, y)$ between two nodes $x$ and $y$ is defined as the length of the shortest path between

$x$ and $y$. For simple graphs, this provides a natural metric space structure, and therefore all topological concepts of metric spaces take a precise signification: sphere, ball, etc. In this respect, two characteristic numbers can be considered. The first $d_1$ is defined by the asymptotic law behaviour (if any) of the volume $A(l)$, of a *ball* of radius $l$ ($l$ = integer):

$$d_1 = \lim_{l\to\infty} \ln A(l)/\ln l. \tag{1}$$

The second is defined similarly from the asymptotic behaviour of the 'surface' $C(l)$ of a *sphere* of radius $l$:

$$d_2 = 1 + \lim_{l\to\infty} (\ln C(l)/\ln l). \tag{2}$$

For infinite lattice graphs, the average over all possible origins is to be taken in (1), (2). As defined above, $A(l)$ (resp. $C(l)$) is the number of lattice sites whose shortest path to a fixed origin consists of $n$ bonds, with $n \leq l$ (resp. $n = l$). As was pointed out by several authors (Kasteleyn 1963, McKenzie 1981, Suzuki 1983), $d_1$ and $d_2$ as defined by (1), (2) are expected to be different in general, and also be distinct from the fractal dimension $\bar{d}$ of a fractal lattice (Suzuki 1983)).

In the case of standard Euclidean lattices (simple cubic for instance), it is easy to show that $d_1 = d_2 = d$. This double equality is a simple consequence of the known equivalence between different metrics on Euclidean spaces. On a fractal lattice, $d_1$ and $d_2$ may be different from $\bar{d}$. For instance, for the von Koch curve we have $d_1 = d_2 = 1$ and $\bar{d} = \ln 4/\ln 3$. The same hold also for a linear polymer and for a random walk trajectory. In both these particular cases, we have $d_1 = d_2 = \tilde{d} = 1$. Moreover, it is clear that both $d_1$, $d_2$ and $\tilde{d}$ are intrinsic parameters dealing with graph theory concepts (paths, loops, ...). A natural question arises: is $c_1$ (or $d_2$) equal to $\tilde{d}$ on a general fractal lattice? In the following, we shall give a negative answer to this question, by studying carefully two examples of fractal lattices: the 2D Sierpenski gasket ($\bar{d} = \ln 3/\ln 2$, $\tilde{d} = 2\ln 3/\ln 5$) and the infinite percolation clusters in two dimensions.

In figure 1 are shown the results obtained for $C(l)$ as a function of the radius $l$ for the 2D Sierpenski gasket. $C(l)$ represents the average over all possible origins on the gasket (exact enumeration) for $1 \leq l \leq 96$. As can be seen, this plot exhibits a self-similar behaviour and $C(l)$ does not behave as a simple power law as a function of $l$. Monte Carlo calculations for $l \geq 96$ show similar results and $d_2$ (equation (2)) cannot be defined simply. Of course $C(l)$ may be smoothed or bounded by power laws of $l$. Such a procedure will provide a possible determination of the value of $d_2$, but will not be discussed further here. Figure 2 gives the corresponding results for $A(l)$ in the same range of $l$: $1 \leq l \leq 96$. $A(l)$ behaves like a proper power law of $l$, and the value $d_1$ of the corresponding exponent was obtained as $d_1 = 1.58 + 0.02$. These results answer our above questions and provide a non-trivial example where $d_1 > \tilde{d}$. More precisely it can be shown that $d_1 = \bar{d}$ on the Sierpinski gaskets. This statement is a direct consequence of the equivalence between the used metric on the gasket (induced by its graph structure) and that of the triangular (at $d = 2$) lattice. This result is easy to obtain and holds at $d = 3, 4, \ldots$. In general, the fractal dimension $\bar{d}$ provides an obvious upper bound for $d_1$: $d_1 \leq \bar{d}$. The Sierpinski gaskets give therefore an example where this upper bound is reached. On the contrary, the example of the von Koch curve shows that $d_1 = \tilde{d}(-1)$. Therefore, it is tempting to conjecture the following inequality:

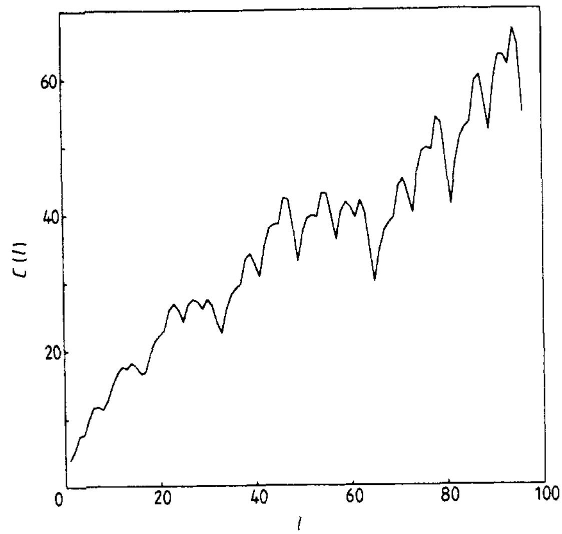


**Figure 1.** The graph of the average 'surface' $C(l)$ of a sphere of radius $l$ on a 2D Sierpinski gasket (SG) $1 \leq l \leq 96$. The self similarity of this curve is clearly shown.

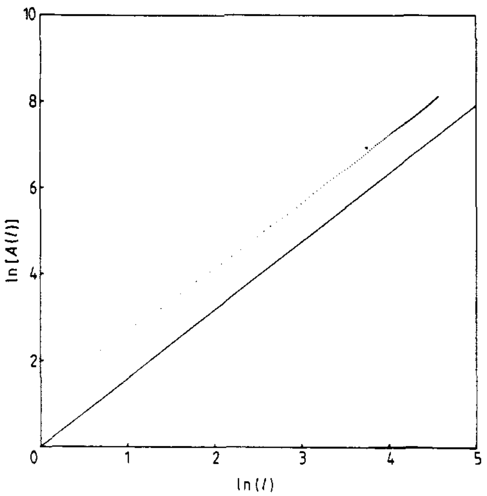


**Figure 2.** Log–log plot of the average volume $A(l)$ of a ball of radius $l$ for a 2D SG. The full straight line has the slope $\bar{d} = \ln 3/\ln 2$ (fractal dimension).

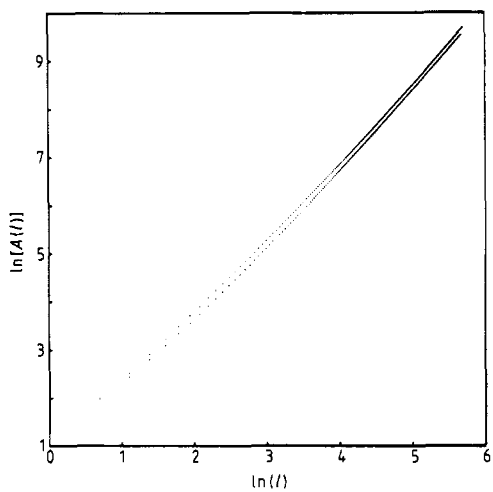


**Figure 3.** Log–log plot of the average volume $A(l)$ of a ball of radius $l$, for an infinite site percolation cluster, at threshold $p_c$. Upper curve: triangular lattice ($p_c = 0.5$). Lower curve: square lattice ($p_c = 0.5927$). The asymptotic slope $\hat{d} = 1.72 \pm 0.02$ is the same in these two cases.

$\tilde{d} \leq d_l \leq \bar{d}$ on a general fractal lattice. Besides the above examples, this conjecture is supported by the results obtained for the percolation clusters.

We have performed the calculations of both $A(l)$ and $C(l)$ on the infinite site percolation clusters at threshold $p_c$. A standard Monte Carlo procedure was used to generate clusters for the square lattice ($p_c = 0.5927$) and the triangular lattice ($p_c = 0.5$). Statistics over some 1500 clusters, of lateral extension $L \geq 600$, were performed in each case. The asymptotic behaviour is reached at $l \geqslant 10^2$ for both $A(l)$ and $C(l)$. In figure 3, we show the results obtained for $A(l)$. The exponent $d_1$ governing the power law of $A(l)$ was found to be $d_1 = 1.72 \pm 0.02$. This value is the same on the square lattice and the triangular lattice. The same number (up to our accuracy) was obtained for $d_2$, from the data relative to $C(l)$.

These results suggest that $d_1 = d_2$ on the percolation clusters. The numerical value of this exponent is therefore universal and does not depend on the detailed structure of the lattice (square or triangular). In the following this exponent will be denoted $\hat{d}$. Moreover the numerical estimate of $\hat{d}$ supports fully our conjecture: $\tilde{d} \leqslant \hat{d} \leqslant \bar{d}$ on the 2D percolation clusters where $\bar{d} = 1.896$ and $\tilde{d} \simeq 4/3$. The double equality is obviously reached at $d = 1$ where $p_c = 1$ and $\tilde{d} = \hat{d} = \bar{d} = 1$. However, it is not clear for us if $\hat{d}$ is related to the usual critical exponents ($\beta_p$, $\nu_p$, ...) of the percolation transition, in a simple way like $\bar{d}$ and $\tilde{d}$.

To conclude, we have presented in this letter a summary of some results relative to the connectivity dimension of fractal lattices. Our results suggest that in general this new intrinsic parameter $\hat{d}$ lies between the fractal $\bar{d}$ and spectral $\tilde{d}$ dimensions of the considered structure: $\tilde{d} \leqslant \hat{d} \leqslant \bar{d}$. The upper and lower bounds can be reached in particular situations. We have shown that $\hat{d}$ is a universal exponent on percolation clusters in two dimensions ($d = 2$). Results pertaining to the percolation clusters at $d = 3, 4, 5, 6$ will be presented elsewhere (Angles d'Auriac *et al* 1984). The relevance of this new intrinsic exponent $\hat{d}$ for various statistical problems deserves a careful investgation in the furture. Potential examples are the SAWS, dimer problems, etc.

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